B.Sc. part-II (Sub) Diff. Calculus 08-07-2020.

Successive Diff.

Q.1. If $y = \log\left(\frac{x}{a+bx}\right)^x$, prove that

$$x^3 y_2 = (y - xy_1)^2$$

Soln. We have, $y = \log\left(\frac{x}{a+bx}\right)^x$

$$\Rightarrow y = x \log \frac{x}{a+bx} \quad \text{——— (1)}$$

$$\Rightarrow y = x[\log x - \log(a+bx)] = x\log x - x\log(a+bx)$$

Diff. both sides with respect to x, we get

$$y_1 = 1\cdot\log x + x\cdot\frac{1}{x} - 1\cdot\log(a+bx) - \frac{bx}{a+bx}$$

$$= [\log x - \log(a+bx)] + \left[1 - \frac{bx}{a+bx}\right]$$

$$\Rightarrow y_1 = \log\frac{x}{a+bx} + \frac{a+\cancel{bx}-\cancel{bx}}{a+bx}$$

$$\Rightarrow y_1 = \frac{y}{x} + \frac{a}{a+bx} \quad \text{——— (2)} \quad \left[\because \frac{y}{x} = \log\frac{x}{a+bx}\right]$$

Again, diff. both sides w.r. to x, we get

$$y_2 = \frac{xy_1 - y}{x^2} - \frac{ab}{(a+bx)^2}$$

Multiplying both sides by x^3, we get

$$x^3 y_2 = x(xy_1 - y) - \frac{abx^3}{(a+bx)^2} \quad \text{——— (3)}$$

From (2), $y_1 - \frac{y}{x} = \frac{a}{a+bx}$

$$\Rightarrow xy_1 - y = \frac{ax}{a+bx}$$

Putting this value in (3), we get

$$x^3 y_2 = \frac{ax^2}{a+bx} - \frac{ab\,x^3}{(a+bx)^2}$$

$$= \frac{ax^2(a+\cancel{bx}-\cancel{bx})}{(a+bx)^2} = \frac{a^2x^2}{(a+bx)^2}$$

$$= \left(\frac{ax}{a+bx}\right)^2 = (xy_1 - y)^2 = (y - xy_1)^2.$$

——— proved.

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Successive Differentiation

Problem based on the applications of Leibnitz's Theorem

Q. 8. If $y = A\,(x+\sqrt{x^2-1})^n + B\,(x-\sqrt{x^2-1})^n$, prove that

$$(x^2-1)y_2 + xy_1 - n^2y = 0$$

And deduce that $(x^2-1)\,y_{n+2} + x(2n+1)\,y_{n+1} = 0$

Soln: We have, $y = A\,(x+\sqrt{x^2-1})^n + B\,(x-\sqrt{x^2-1})^n$

Diff. both sides with respect to x, we get

$$y_1 = nA(x+\sqrt{x^2-1})^{n-1}\left(1+\frac{1}{2\sqrt{x^2-1}}\times 2x\right)$$
$$+ nB\,(x-\sqrt{x^2-1})^{n-1}\left(1-\frac{1}{2\sqrt{x^2-1}}\,2x\right)$$

$$\therefore y_1 = nA\,(x+\sqrt{x^2-1})^{n-1}\left(\frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}}\right)$$
$$- nB\,(x-\sqrt{x^2-1})^{n-1}\left(\frac{x-\sqrt{x^2-1}}{\sqrt{x^2-1}}\right)$$

$$\Rightarrow \sqrt{x^2-1}\;y_1 = nA\,(x+\sqrt{x^2-1})^n - nB\,(x-\sqrt{x^2-1})^n$$

Again, diff. both sides, w.r. to x, we get

$$\sqrt{x^2-1}\,y_2 + \frac{1}{2\sqrt{x^2-1}}\,2xy_1$$
$$= n^2A\,(x+\sqrt{x^2-1})^{n-1}\left(1+\frac{1}{2\sqrt{x^2-1}}\,2x\right)$$
$$- n^2B\,(x-\sqrt{x^2-1})^{n-1}\times\left(1-\frac{1}{2\sqrt{x^2-1}}\,2x\right)$$

$$\therefore (x^2-1)y_2 + xy_1 = n^2A\,(x+\sqrt{x^2-1})^n + n^2B\,(x-\sqrt{x^2-1})^n$$

$$\Rightarrow (x^2-1)y_2 + xy_1 = n^2\left[A(x+\sqrt{x^2-1})^n + B(x-\sqrt{x^2-1})^n\right]$$

Hence, $(x^2-1)y_2 + xy_1 - n^2y = 0$

Differentiating this n times w.r. to x, by Leibnitz's theorem, we get

$$y_{n+2}(x^2-1) + {}^nC_1\,y_{n+1}(2x) + \frac{n(n-1)}{1.2}\,y_n(2) + xy_{n+1} + {}^nC_1\,y_n$$
$$- n^2y_n = 0$$

$$\Rightarrow (x^2-1)\,y_{n+2} + x(2n+1)\,y_{n+1} = 0$$

proved