

Q. 1. If $y = \log \left(\frac{x}{a+bx} \right)^x$, prove that

$$x^3 y_2 = (y - xy_1)^2$$

Soln. We have, $y = \log \left(\frac{x}{a+bx} \right)^x$

$$\Rightarrow y = x \log \frac{x}{a+bx} \quad \text{--- (1)}$$

$$\Rightarrow y = x [\log x - \log(a+bx)] = x \log x - x \log(a+bx)$$

Diff. both sides with respect to x , we get

$$\begin{aligned} y_1 &= 1 \cdot \log x + x \cdot \frac{1}{x} - 1 \cdot \log(a+bx) - \frac{bx}{a+bx} \\ &= [\log x - \log(a+bx)] + \left[1 - \frac{bx}{a+bx} \right] \end{aligned}$$

$$\Rightarrow y_1 = \log \frac{x}{a+bx} + \frac{a+bx-bx}{a+bx}$$

$$\Rightarrow y_1 = \frac{y}{x} + \frac{a}{a+bx} \quad \text{--- (2)} \quad \left[\because \frac{y}{x} = \log \frac{x}{a+bx} \right]$$

Again, diff. both sides w.r. to x , we get

$$y_2 = \frac{xy_1 - y}{x^2} - \frac{ab}{(a+bx)^2}$$

Multiplying both sides by x^3 , we get

$$x^3 y_2 = x(xy_1 - y) - \frac{abx^3}{(a+bx)^2} \quad \text{--- (3)}$$

From (2), $y_1 - \frac{y}{x} = \frac{a}{a+bx}$

$$\Rightarrow xy_1 - y = \frac{ax}{a+bx}$$

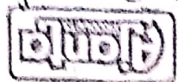
Putting this value in (3), we get

$$x^3 y_2 = \frac{ax^2}{a+bx} - \frac{abx^3}{(a+bx)^2}$$

$$= \frac{ax^2(a+bx-bx)}{(a+bx)^2} = \frac{a^2x^2}{(a+bx)^2}$$

$$= \left(\frac{ax}{a+bx} \right)^2 = (xy_1 - y)^2 = (y - xy_1)^2$$

proved.



Successive Differentiation

Problem based on the applications of Leibnitz's Theorem.

Q.2. If $y = A(x + \sqrt{x^2 - 1})^n + B(x - \sqrt{x^2 - 1})^n$, prove that

$$(x^2 - 1)y_2 + xy_1 - ny = 0$$

$$\text{And deduce that } (x^2 - 1)y_{n+2} + x(2n+1)y_{n+1} - n^2y = 0$$

Soln. We have, $y = A(x + \sqrt{x^2 - 1})^n + B(x - \sqrt{x^2 - 1})^n$

Diff. both sides with respect to x , we get

$$y_1 = nA(x + \sqrt{x^2 - 1})^{n-1} \left(1 + \frac{1}{2\sqrt{x^2 - 1}} \cdot 2x\right) + nB(x - \sqrt{x^2 - 1})^{n-1} \left(1 - \frac{1}{2\sqrt{x^2 - 1}} \cdot 2x\right)$$

$$\therefore y_1 = nA(x + \sqrt{x^2 - 1})^{n-1} \left(\frac{x + \sqrt{x^2 - 1}}{\sqrt{x^2 - 1}}\right) - nB(x - \sqrt{x^2 - 1})^{n-1} \left(\frac{x - \sqrt{x^2 - 1}}{\sqrt{x^2 - 1}}\right)$$

$$\Rightarrow \sqrt{x^2 - 1} y_1 = nA(x + \sqrt{x^2 - 1})^n - nB(x - \sqrt{x^2 - 1})^n$$

Again, diff. both sides w.r. to x , we get

$$\begin{aligned} \sqrt{x^2 - 1} y_2 + \frac{1}{2\sqrt{x^2 - 1}} \cdot 2x y_1 &= n^2 A(x + \sqrt{x^2 - 1})^{n-1} \left(1 + \frac{1}{2\sqrt{x^2 - 1}} \cdot 2x\right) \\ &\quad - n^2 B(x - \sqrt{x^2 - 1})^{n-1} \left(1 - \frac{1}{2\sqrt{x^2 - 1}} \cdot 2x\right) \end{aligned}$$

$$\therefore (x^2 - 1)y_2 + xy_1 = n^2 A(x + \sqrt{x^2 - 1})^n + n^2 B(x - \sqrt{x^2 - 1})^n$$

$$\Rightarrow (x^2 - 1)y_2 + xy_1 = n^2 [A(x + \sqrt{x^2 - 1})^n + B(x - \sqrt{x^2 - 1})^n]$$

$$\text{Hence, } (x^2 - 1)y_2 + xy_1 - n^2y = 0$$

Differentiating this n times w.r. to x , by Leibnitz's theorem, we get

$$y_{n+2} (x^2 - 1) + \frac{n}{1} y_{n+1} (2x) + \frac{n(n-1)}{1 \cdot 2} y_n (2) + y_{n+1} \cdot x + \frac{n}{1} y_1 - n^2 y_n = 0$$

$$\Rightarrow (x^2 - 1)y_{n+2} + x(2n+1)y_{n+1} - n^2y = 0$$

proved